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Orthogonal Polynomial Solutions of Even-Ordered Ordinary Differential Equations

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INTRODUCTION

There is associated with least-squares approximation a vast literature concerning orthogonal functions, mainly because such functions furnish a convenient system of coordinates in function spaces. Among the most extensively studied orthogonal systems are systems of orthogonal polynomials. However, it should be noted that the use of orthogonal polynomials as approximating functions is not limited to L_2-space. See, for instance, their use in [2, 5].

A standard method for the development of the theory of classical orthogonal polynomials such as the Legendre, Hermite, and Laguerre polynomials has been treating them as solutions of certain second order differential equations. In this paper we show that each of these sets of classical polynomials can be obtained as the solutions of an infinite set of even-ordered differential equations.

Brenke [1] and Meux [4] have obtained conditions for the existence of orthogonal polynomial solutions of second- and fourth-order ordinary differential equations, respectively. They showed that with the proper choice of a weight function and a proper choice of the interval of orthogonality, the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite are obtained.

This paper uses the results of Krall [3] concerning self-adjoint differential expressions in order to obtain results for general even-ordered ordinary differential equations.

POLYNOMIAL SOLUTIONS

Consider the differential equation

$$\sum_{k=0}^{2n-1} Q_{2n-k} y_i^{(2n-k)} + \lambda_i y_i = 0. \tag{1}$$

It can be shown in order for this equation to have polynomial solutions of the form

$$y_i = \sum_{j=0}^{i} a_{ji} x^{i-j}, \qquad a_{0i} \neq 0, \tag{2}$$

that each coefficient $Q_{2n-k}(x)$ must be a polynomial of degree at most $2n - k$ (see [1]). The details of the proof are completely analogous to those in [1] and are omitted here.

Self-Adjoint Property

Krall [3] has shown that the most general self-adjoint differential expression of order $2n$ is

$$\sum_{s=0}^{n} A_s y^{(2s)} + \sum_{s=1}^{n} \sum_{k=0}^{s-1} \binom{2s}{2k+1} \frac{2^{2s-2k} - 1}{s-k} B_{2s-2k} A_s^{(2s-2k-1)} y^{(2k+1)}, \tag{3}$$

where the B_i's are the Bernoulli numbers. Also, if we use $B_1 = \frac{1}{2}$, $0 = B_3 = B_5 = \cdots$, we can write (3) as

$$\sum_{s=0}^{n} \sum_{k=0}^{2s} (-1)^{k+1} \binom{2s}{k} \frac{2^{2s-k+1} - 1}{2s-k+1} 2B_{2s-k+1} A_s^{(2s-k)} y^{(k)}. \tag{4}$$

The following theorem gives sufficient conditions for the self-adjointness of Eq. (1) over (a, b) with respect to a suitably chosen weight function $W(x)$.

Theorem 1. *If $W(x) \in C^{2n}$ on (a, b) and if*

$$WQ_{2k+1} = \sum_{s>k}^{n} \binom{2s}{2k+1} \frac{2^{2s-2k} - 1}{s-k} B_{2s-2k} (WQ_{2s})^{(2s-2k-1)},$$

$$k = 0, 1, \ldots, n-1, \tag{5}$$

then each equation of the family (1), after multiplication by $W(x)$, is self-adjoint.

Proof. The result follows immediately upon the multiplication of (1) by $W(x)$ and applying (3), which is the most general self-adjoint differential expression of order $2n$.

ORTHOGONALITY

In order to apply the standard technique for showing the orthogonality of the solution set of (1), it is necessary to rewrite the self-adjoint differential equation (5) in the form

$$\sum_{s=0}^{n} [T_s(x)\, y_i^{(s)}]^{(s)} = 0, \tag{6}$$

where the T_i's are to be determined.

From (4) the coefficients of each $y_i(k)$ are of the form

$$\sum_{s>[(k-1)/2]}^{n} (-1)^{k+1} \binom{2s}{k} \frac{2^{2s-k+1}-1}{2s-k+1}\, 2B_{2s-k+1}(WQ_{2s})^{(2s-k)},$$
$$k = 0, 1,..., 2n. \tag{7}$$

Expanding (6) by Leibnitz's rule for higher ordered derivatives and regrouping the coefficients of each $y_i^{(k)}$, we conclude that the coefficients are given by

$$\sum_{i=0}^{[k/2]} \binom{k-[k/2]+i}{[k/2]-i} T_{k-[k/2]+i}^{(k-2[k/2]+2i)}, \qquad k = 0, 1,..., 2n, \tag{8}$$

where $[k/2]$ is the usual greatest-integer-function and T_i is defined to be identically zero for $i > n$.

In order to solve for the T_i's, we equate (7) and (8) and consider $k = 2n, 2n-2,..., 2$, obtaining in turn the values of T_n, T_{n-1},..., T_1. Now if the techniques used in [4] are applied $n-1$ times, we obtain

$$\int_a^b Wy_iy_j\, dx = 0.$$

That is, the solutions of (1) are orthogonal on (a, b) with respect to $W(x)$. We state this result as the following theorem.

THEOREM 2. *If the conditions of Theorem* 1 *are satisfied and if*

$$T_s^{(k)} \doteq 0 \quad at \quad x = a \quad and \quad x = b \qquad (0 \leqslant k < s \leqslant n)$$

then for $\lambda_i \neq \lambda_j$ the corresponding solutions of (1) *form an orthogonal set of polynomials on $a \leqslant x \leqslant b$ with respect to the positive weight function $W(x)$.*

An Example. Consider now the sixth-order differential equation and suppose that $W(x) \equiv 1$ on the interval $(-1, 1)$. By using (5), equating (7) and (8), and using methods analogous to those in [4], we obtain the self-adjoint differential equation

$$[(1 - x^2)^3 y''']''' + [8(x^2 - 1)^2 y'']'' - [4(x^2 - 1) y']' + \lambda^3(\lambda + 1)^3 y = 0, \tag{9}$$

which has the classical Legendre polynomials as solutions. This equation could very well be called the sixth-order Legendre differential equation. It is interesting to note the similarity of the coefficients of the various $y^{(k)}$'s in (9) and the coefficients of the classical second order Legendre differential equation

$$(1 - x^2) y'' - 2xy' + \lambda(\lambda + 1) y = 0,$$

and those given in [4] by

$$[(1 - x^2)^2 y'']'' + [2(x^2 - 1) y']' - \lambda^2(\lambda + 1)^2 y = 0.$$

Also, similar results can be obtained for the Jacobi, Laguerre, and Hermite polynomials. Thus, an infinite set of even-ordered differential equations has been found to be associated with each of these classical orthogonal polynomials.

Many interesting questions arise from these results. For example, what is the general form of the Legendre differential equation of order $2n$? Can these higher ordered classical differential equations be obtained by the use of the usual generating functions for the polynomials ?

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